Transitive Openings

D. Boixader[1] **J. Recasens**[2]

[1]UPC, Spain, j.recasens@upc.edu
[2]UPC, Spain, dionis.boixader@upc.edu

Abstract

Though the transitive closure of a reflexive and symmetric fuzzy relation R is unique and there are several algorithms to calculate it, there can be many transitive openings (maximal T-indistinguishability operators among the ones smaller than or equal to R). This paper presents a method to calculate transitive openings of a reflexive and symmetric fuzzy relation R. It is worth noticing that apart form the minimum t-norm, this is the first algorithm that allows us to calculate them.

Keywords: Similarity, Indistinguishability Operator, Finite-Valued t-norm, Transitive Closure, Transitive Opening

1. Introduction

The transitive closure of a reflexive and symmetric fuzzy relation R gives a T-indistinguishability operator greater than or equal to R. In this case it is possible to obtain the best upper approximation since the infimum of T-indistinguishability operators is also a T-indistinguishability operator. If we want a lower approximation, then the situation is more complicated since the supremum of indistinguishability operators is not such an operator in general. What we can find is T-indistinguishability operators maximal among the ones that are smaller than or equal to a given reflexive and symmetric fuzzy relation. These relations are called transitive openings and in general they are not unique, but there can be an infinite quantity of them, even in sets of finite cardinality.

There is no general method in the literature to calculate them. In [5], an algorithm to find maximal transitive openings of a given fuzzy relation is given, but the obtained openings are not symmetric in general and in the process of symmetrizing them, maximality can be lost. Heuristic methods to obtain T-indistinguishability operators smaller than or equal to a given fuzzy relation close to maximal ones have been proposed [3], but until now there is not a general methodology to find them.

The minimum t-norm is an exception because of the special behaviour of min-indistinguishability operators. In this case there are a number of algorithms to find at least some of the min-transitive openings of a given reflexive and symmetric fuzzy relation. A classic method is the complete linkage. Other algorithms can be found in [4] [6].

In this presentation a method of calculating all the transitive openings of a reflexive and symmetric L-relation on a finite set X will be explained. This result will then be exploited to obtain transitive openings with respect to the Łukasiewicz t-norm of a given reflexive and symmetric fuzzy relation (valued in the unit interval). The method is very effective when combined with heuristic algorithms such as the ones proposed in [3] (see Example 3.8).

2. Preliminaries

This section contains some definitions and results on finite-valued t-norms that will be needed later on the paper. The proofs of the results as well as more information about finite-valued t-norms can be found in [11]. The definitions of indistinguishability operator, proximity and transitive closure are also recalled.

The study of operators defined on a finite chain L is of great interest, especially because reasoning is usually done by using linguistic terms or labels that are totally ordered. For instance, the size of an object can be granularized in *very small, small, average, big, very big.* If an operator T is defined on this set, then we will be able to combine these labels in order to obtain for example $T(average, very\ big)$. The calculations are simplified greatly by addressing the problem of combining labels in this way, since there is no need to assign numerical values to them or to identify them with an interval or with a fuzzy subset.

Finite chains are also useful in cases in which the values are discrete by nature or by discretization. On a customer-satisfaction survey, respondents may be asked to describe their satisfaction with a service using natural numbers from 0 to 5 or labels ranging from *not at all satisfied* to *very satisfied.*

In this line, various authors have translated t-norms and t-conorms to finite chains ([10], [11]) obtaining interesting theoretical results.

Let L be a finite totally ordered set with minimum e and maximum u.

Definition 2.1. *A binary operation $T : L \times L \to L$ is a t-norm if and only if for all $x, y, z \in L$*

1. $T(x, y) = T(y, x)$

2. $T(T(x,y),z) = T(x,T(y,z))$
3. $T(x,y) \leq T(x,z)$ whenever $y \leq z$
4. $T(x,u) = x$.

The set of t-norms on a finite chain depends only on its cardinality. For this reason we will only consider the chains $L = \{0 = \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$.

Example 2.2.

1. *The Minimum t-norm T on L is defined by* $T(i,j) = \min\{i,j\}$.
2. *The Łukasiewicz t-norm T on L is defined by* $T(i,j) = \max\{i+j-1, 0\}$.

Definition 2.3.

- *A map $f : L \to L$ is smooth if and only if*
$$0 \leq f(i+1) - f(i) \leq \frac{1}{n} \text{ for all } i \in L,\ i < 1.$$
- *A map $F : L \times L \to L$ is smooth if and only if it is smooth with respect to both variables.*

Smooth t-norms on finite chains are the equivalent of continuous ones defined on [0,1].

Definition 2.4. *A t-norm T on L is divisible if and only if for all $i, j \in L$ with $i \leq j$ there exists $k \in L$ such that*
$$i = T(j,k).$$

Smoothness and divisibility are equivalent concepts for t-norms.

Proposition 2.5. *A t-norm on L is smooth if and only if it is divisible.*

The next proposition characterizes all smooth t-norms on L as particular ordinal sums of copies of the t-norm of Łukasiewicz.

Proposition 2.6. *A t-norm T on L is smooth if and only if there exists $J = \{0 = i_0 < i_1 < ... < i_m = 1\} \subseteq L$ such that*

$$T(i,j) = \begin{cases} \max\{i_k, i+j-i_k\} & \text{if } i,j \in [i_k, i_{k+1}] \\ & \text{for some } i_k \in J \\ \min\{i,j\} & \text{otherwise.} \end{cases}$$

T is said to be an ordinal sum and is represented by $T =< 0 = i_0, i_1, ...i_m = 1 >$.

Indistinguishability operators fuzzify the concepts of crisp equality and crisp equivalence relation. They have been studied under different settings, mainly valued on [0,1] and with respect to a left continuous t-norm, though some generalizations to more general structures like GL-monoids have been carried on.

Definition 2.7.

- *A fuzzy relation is a map $X \times X \to [0,1]$*
- *An L-relation is a map $X \times X \to L$.*

Definition 2.8. *Given a t-norm T, a T-indistinguishability operator (L-indistinguishability operator) E on a set X is a fuzzy relation (an L-relation) on X satisfying for all $x, y, z \in X$*

1. *$E(x,x) = 1$ (Reflexivity)*
2. *$E(x,y) = E(y,x)$ (Symmetry)*
3. *$T(E(x,y), E(y,z)) \leq E(x,z)$ (T-transitivity).*

Definition 2.9. *A fuzzy relation (L-relation) R on a set X is a proximity relation if and only if it is reflexive and symmetric.*

Definition 2.10. *Let T be a t-norm on L (on $[0,1]$) and R a proximity relation on a set X. The T-transitive closure of R is the L-indistinguishability operator (T-indistinguishability operator) E on X satisfying*

1. *$R \leq E$.*
2. *If E' is an L-indistinguishability operator (T-indistinguishability operator) on X such that $R \leq E' \leq E$, then $E' = E$.*

3. Transitive Openings

In this section a method for calculating all the transitive openings of a reflexive and symmetric L'-relation on a finite set X will be explained. This result will then be exploited to obtain transitive openings with respect to the Łukasiewicz t-norm of a given reflexive and symmetric fuzzy relation (valued in the unit interval). The method is very effective when combined with heuristic algorithms such as the ones proposed in [3] (see Example 3.8).

Definition 3.1. *Let R be a proximity relation on a set X and T a t-norm on L. A T-indistinguishability operator (L-indistinguishability operator) $\underline{R}$ on X is a T-transitive opening (L-transitive opening) of R if and only if*

- *$\underline{R} \leq R$*
- *If E is another T-indistinguishability operator (L-indistinguishability operator) on X satisfying $E \leq R$, then $E \leq \underline{R}$.*

Proposition 3.2. *Let R be a proximity relation on a finite set $X = \{r_1, r_2, ..., r_s\}$ of cardinality s and T a t-norm. S is a T-indistinguishability operator (L-indistinguishability operator) smaller than or equal to R if and only if its entries satisfy the following system of inequalities:*

$$\begin{aligned} 0 \leq S(r_i, r_j) &\leq R(r_i, r_j) \\ &\text{for all } i,j = 1,2,...,s. \\ T(S(r_i,r_j), S(r_j,r_k)) &\leq S(r_i,r_k) \\ &\text{for all } i,j,k = 1,2,...,s. \\ S(r_i,r_j) &= S(r_j,r_i) \\ &\text{for all } i,j = 1,2,...,s. \end{aligned}$$

Proof. Trivial. □

Example 3.3. *Let us consider the reflexive and symmetric L-relation R on $X = \{a, b, c\}$ with $L = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$.*

$$R = \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & \frac{2}{3} \\ 0 & \frac{2}{3} & 1 \end{pmatrix}.$$

An L-relation S on X with matrix

$$S = \begin{pmatrix} 1 & p & q \\ p & 1 & r \\ q & r & 1 \end{pmatrix}$$

is an L-indistinguishability operator smaller than or equal to R if and only if

$$\begin{aligned} 0 \le p &\le \frac{2}{3} \\ 0 \le q &\le 0 \\ 0 \le r &\le \frac{2}{3} \\ T(p, q) &\le r \\ T(p, r) &\le q \\ T(q, p) &\le r \\ T(q, r) &\le p \\ T(r, p) &\le q \\ T(r, q) &\le p. \end{aligned}$$

If T is the t-norm of Łukasiewicz, then there are 8 possible solutions:

$$p = 0, \frac{1}{3}, \quad q = 0, \quad r = 0, \frac{1}{3}, \frac{2}{3}$$
$$p = \frac{2}{3}, \quad q = 0, \quad r = 0, \frac{1}{3}.$$

Among them, there are 2 L-transitive openings of R. Namely

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & \frac{2}{3} & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 \end{pmatrix}.$$

This exemplifies how the transitive openings of a reflexive and symmetric L-relation can be obtained. The method is very greedy and in general would need many calculations. Fortunately, better lower bounds can be found for the entries of the matrices. For example, it is well known ([15]) that the infimum of the L-indistinguishability operators generated by the columns of a reflexive and symmetric L-relation R is always smaller than or equal to R. In the previous example, this infimum is

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 \end{pmatrix}$$

and hence the first three inequalities can be replaced by

$$\begin{aligned} \frac{1}{3} &\le p \le \frac{2}{3} \\ 0 &\le q \le 0 \\ \frac{1}{3} &\le r \le \frac{2}{3}. \end{aligned}$$

Better lower bounds can be found using existing heuristic methods (cf [3], for example) to calculate lower approximations of reflexive and symmetric fuzzy relations by T-transitive ones. This is illustrated by Example 3.8.

The previous results can be applied to obtain transitive openings of reflexive and symmetric fuzzy relations valued in the unit interval as will be proved in Theorem 3.7.

Definition 3.4. *For $\alpha \in [0, 1]$, let $\lfloor \alpha \rfloor$ be the greatest value of L satisfying $\lfloor \alpha \rfloor \le \alpha$.*

Lemma 3.5. *Let R be a reflexive and symmetric L-relation on a finite set X. If $\underline{R}$ is an L-transitive opening of R with respect to the Łukasiewicz t-norm T on L and S is a T-transitive opening of R as a fuzzy relation valued on $[0, 1]$ with $\underline{R} \le S$, then $\underline{R}(x, y) = \lfloor S(x, y) \rfloor$ for all $x, y \in X$.*

Proof. $\lfloor S \rfloor$ is an L-indistinguishability operator greater than or equal to $\underline{R}$. Since $\underline{R}$ is a transitive opening of R, $\underline{R} = \lfloor S \rfloor$. □

Lemma 3.6. *Let $L = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$, R be a reflexive and symmetric L-relation on a finite set X of cardinality s and $\underline{R}$ an L-transitive opening of R with respect to the Łukasiewicz t-norm T. Given $a, b \in X$ such that $\underline{R}(a, b) < R(a, b)$ and $0 < \epsilon < \frac{1}{n}$, let us consider the following fuzzy relation R':*

$$R'(x, y) = \begin{cases} \underline{R}(a, b) + \epsilon & \text{if } (x, y) = (a, b) \\ & \text{or } (y, x) = (a, b) \\ \underline{R}(x, y) & \text{otherwise.} \end{cases}$$

Then the T-transitive closure $\overline{R'}$ of R' is not a transitive opening of R in $[0, 1]$.

Proof. Let us consider the L-relation R'' on X.

$$R''(x, y) = \begin{cases} \underline{R}(a, b) + \frac{1}{n} & \text{if } (x, y) = (a, b) \\ & \text{or } (y, x) = (a, b) \\ \underline{R}(x, y) & \text{otherwise.} \end{cases}$$

Since $\underline{R}$ is a transitive opening of R in L, the transitive closure $\overline{R''}$ of R'' is not smaller than or equal to R. Therefore there exist $x, y \in X$ with $\overline{R''}(x, y) > R(x, y)$.

Since X is finite of cardinality s,

$$\begin{aligned} &\overline{R''}(x, y) \\ &\quad = \max_{z_1, \dots, z_{s-1}} T(R''(x, z_1), R''(z_1, z_2), ..., R''(z_{s-1}, y)) \end{aligned}$$

and this maximum is attained when it contains $R''(a, b)$. Then there exist $z_1, z_2, ..., z_{s-3} \in X$ such that

$$\begin{aligned} &\overline{R''}(x, y) \\ &\quad = T(R''(x, z_1), R''(z_1, z_2), ..., R''(z_i, a), R''(a, b), \\ &\quad R''(b, z_{i+1}), R''(z_{s-3}, y)) \\ &\quad = T(\underline{R}(x, z_1), \underline{R}(z_1, z_2), ..., \underline{R}(z_i, a), \underline{R}(a, b) \\ &\quad + \frac{1}{n}, \underline{R}(b, z_{i+1}), \underline{R}(z_{s-3}, y)). \end{aligned}$$

$$T(\underline{R}(x,a), \underline{R}(a,b) + \frac{1}{n}, \underline{R}(b,y))$$
$$\geq T(R''(x,z_1), R''(z_1,z_2), ..., R''(z_i,a), R''(a,b),$$
$$R''(b,z_{i+1}), R''(z_{s-3},y))$$
$$= \overline{R''}(x,y) > R(x,y)$$

and therefore

$$T(\underline{R}(x,a), \underline{R}(a,b), \underline{R}(b,y)) = \underline{R}(x,y) = R(x,y).$$

So,

$$R'(x,y) \geq T(\underline{R}(x,a), \underline{R}(a,b) + \epsilon, \underline{R}(b,y)) > R(x,y).$$

and therefore $\overline{R'}$ is not a transitive opening of R. □

Theorem 3.7. *Let R be a reflexive and symmetric L-relation on a finite set X of cardinality s and $\underline{R}$ an L-transitive opening of R with respect to the Łukasiewicz t-norm T. Then $\underline{R}$ is also a T-transitive opening of R as a fuzzy relation.*

Proof. Let us consider a fuzzy relation S on X such that
$$\underline{R} < S.$$
Then there exists $a, b \in X$ with $\underline{R}(a,b) < S(a,b)$. The fuzzy relation R' of the previous Lemma 3.6 with $\epsilon \leq S(a,b) - \underline{R}(a,b)$ satisfies

$$\underline{R} < R' \leq S.$$

Since the transitive closure $\overline{R'}$ of R' is greater than R and the map that assigns the transitive closure to a given reflexive and symmetric fuzzy relation is non-decreasing, a fortiori S is greater than R. Therefore there are no T-indistinguishability operators between $\underline{R}$ and R. □

The next example shows the effectiveness of the proposed method for finding transitive openings when combined with heuristic algorithms.

Example 3.8. *Let us consider the reflexive and symmetric fuzzy relation R on a set of cardinality 5 with matrix*

$$R = \begin{pmatrix} 1.0 & 0.5 & 0.5 & 0.7 & 0.6 \\ 0.5 & 1.0 & 0.8 & 0.7 & 0.5 \\ 0.5 & 0.8 & 1.0 & 0.9 & 0.8 \\ 0.7 & 0.7 & 0.9 & 1.0 & 0.9 \\ 0.6 & 0.5 & 0.8 & 0.9 & 1.0 \end{pmatrix}.$$

In [3] an algorithm is used to calculate a T-indistinguishability operator S (T the Lukasiewicz t-norm) which is not a transitive opening of R but smaller than R and close to a transitive opening:

$$S = \begin{pmatrix} 1.0 & 0.5 & 0.5 & 0.6 & 0.6 \\ 0.5 & 1.0 & 0.8 & 0.6 & 0.5 \\ 0.5 & 0.8 & 1.0 & 0.8 & 0.7 \\ 0.6 & 0.6 & 0.8 & 1.0 & 0.9 \\ 0.6 & 0.5 & 0.7 & 0.9 & 1.0 \end{pmatrix}.$$

R (and S) can also be considered L'-relations where $L' = \{0, \frac{1}{10}, \frac{2}{10}, ..., 1\}$. There are only 2 intermediate T-indistinguishability operators valued in L that can be obtained in the same way as after the Example 3.3:

$$\begin{pmatrix} 1.0 & 0.5 & 0.5 & 0.6 & 0.6 \\ 0.5 & 1.0 & 0.8 & 0.6 & 0.5 \\ 0.5 & 0.8 & 1.0 & 0.8 & 0.7 \\ 0.6 & 0.6 & 0.8 & 1.0 & 0.9 \\ 0.6 & 0.5 & 0.7 & 0.9 & 1.0 \end{pmatrix}$$

$$\begin{pmatrix} 1.0 & 0.5 & 0.5 & 0.7 & 0.6 \\ 0.5 & 1.0 & 0.8 & 0.6 & 0.5 \\ 0.5 & 0.8 & 1.0 & 0.8 & 0.7 \\ 0.6 & 0.6 & 0.8 & 1.0 & 0.9 \\ 0.6 & 0.5 & 0.7 & 0.9 & 1.0 \end{pmatrix}.$$

The second matrix is a transitive opening of R as an L-relation and therefore also as a fuzzy relation.

4. Concluding Remarks

A way to obtain all the transitive openings of a reflexive and symmetric L-relation is provided that has been used to obtain transitive openings of a given reflexive and symmetric fuzzy relation.

The algorithm for calculating the transitive openings of reflexive and symmetric fuzzy relations with respect to the t-norm L of Łukasiewicz can be used for any other continuous Archimedean t-norms. Indeed, if T is a continuous Archimedean t-norm with additive generator t and E is a T-indistinguishability operator on a finite set X, then $E' = \alpha - \alpha \circ t \circ E$ is an L-indistinguishability operator on X [2]. In this way we can transfer a transitive opening with respect to the Łukasiewicz t-norm to a transitive opening with respect to another continuous Archimedean t-norm.

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